Wess-Zumino-Witten term

- Donoghue, Golowich, Holstein VII-5
- Weinberg 19.8
- Witten "Global aspects of current algebra"

We have seen how the low energy dynamics of the light mesons are determined by the chiral lagrangian

$$S = \frac{f_\pi^2}{4} \int d^4x \; tr(\partial_\mu U^\dagger \partial^\mu U)$$

Besides the chiral symmetry, this lagrangian has several discrete symmetries:

1) Charge conjugation, $C : U \to U^*$
2) "Naive parity", $P_0 : \vec{x} \to -\vec{x},\ t \to t,\ U \to U$
3) extra $U \to U^\dagger$.

In terms of meson fields, the $U \to U^\dagger$ acts as $\pi^a \to -\pi^a$, since $U = e^{i\frac{\pi^a T^a}{F}}$.

Therefore, this counts the number of pion fields mod 2.

That there are extra symmetries in the IR theory is natural. As we discussed, this is the essence of what accidental symmetries are.

However, experimentally, $\pi \to -\pi$ is badly broken. π^0 decays to $\gamma\gamma$, ϕ meson decays to both $K\bar{K}$ and $\pi\pi\pi$, K decays to both $\pi\pi$ and $\pi\pi\pi$, etc. All "forbidden" processes are in fact large and clearly there is no $\pi \to -\pi$ selection rule.

One can get convinced that this selection rule remains even if higher derivative terms are included.

Still, it turns out we did not write all terms compatible with the chiral $SU(3) \times SU(3)$ symmetry.

Take the fields in the chiral lagrangian to approach a common limit as $x^\mu \to \infty$. Then, we can think of (Euclidean) spacetime as a sphere S_4.

Take now a 5-dimensional ball B_5 with coordinates y^M, or x^μ and s, so that its surface is S_4.
Now extend the pion fields $U(x)$ to $U(y)$.
In 5d, now we can write the action

$$S = \frac{f_\pi^2}{4} \int d^4x \, tr(\partial_\mu U^\dagger \partial^\mu U) + n \int_{B_5} d^5y \, \omega(y)$$

with

$$\omega(y) = -\frac{i}{240\pi^2} \epsilon_{IJKLM} \, tr\left(U^{-1}\frac{\partial U}{\partial y^I} U^{-1}\frac{\partial U}{\partial y^J} U^{-1}\frac{\partial U}{\partial y^K} U^{-1}\frac{\partial U}{\partial y^L} U^{-1}\frac{\partial U}{\partial y^M} \right)$$

This is manifestly $SU(3)_L \times SU(3)_R$ invariant.
Second, due to the ϵ-tensor, the

integral is independent of the choice of 5-dimensional coordinates y^M.

Moreover, the integral only depends on the values of $U(y)$ on the ball's surface. By taking a variation $\delta U(y)$,

$$\delta\left(U^{-1}\frac{\partial U}{\partial y}\right) = -U^{-1}\delta U\, U^{-1}\frac{\partial U}{\partial y} + U^{-1}\frac{\partial \delta U}{\partial y}$$

$$= U^{-1}\frac{\partial}{\partial y}\left(\delta U\, U^{-1}\right) U$$

This leads to a total ∂. When acting on the other terms, you get zero due to ϵ-tensor. Therefore,

$$\delta w(y) = -\frac{i}{48\pi^2}\epsilon^{IJKLM}\frac{\partial}{\partial y^M} tr\left(U^{-1}\partial_I U\, U^{-1}\partial_J U\, U^{-1}\partial_K U\, U^{-1}\partial_L U\, U^{-1}\delta U\right)$$

so a change of $U(y)$ does not affect its value in spacetime. We can therefore write

$$S_{WZW} \supset n \int_{B_5} d^5y \, \omega(y)$$

it is called the Wess-Zumino-Witten action and, while written in 5d it affects physics in 4d only.

Writing U in terms of meson fields,

$$U^\dagger \partial_\mu U = \frac{2i}{f} \partial_\mu \pi + \dots$$

one gets

$$\omega \to \frac{2}{15\pi^2 f_\pi^5} \epsilon_{IJKLM} \, tr\left(\partial_I \pi \, \partial_J \pi \, \partial_K \pi \, \partial_L \pi \, \partial_M \pi\right) + \dots$$

given that this is a total derivative, using stoke's theorem this can be written as

$$S \supset n \frac{2}{15\pi^2 f_\pi^5} \int d^4x \, \epsilon^{\mu\nu\alpha\beta} \, tr\left(\pi \, \partial_\mu \pi \, \partial_\nu \pi \, \partial_\alpha \pi \, \partial_\beta \pi\right) + \dots$$

This is now in 4 dimensions, but chiral symmetry is no longer explicit.

If kaons & pions were massless, this gives the zero-momentum $KK \to \pi\pi\pi$ scattering amplitude.

This term breaks both the $(-1)^{\pi}$ symmetry and the naive parity $\vec{x} \to -\vec{x}$, $t \to t$, $U \to U$.

The symmetry of the combined action

$$S = \frac{f^2}{4} \int d^4x \, tr(\partial_\mu U^\dagger \partial^\mu U) + n \int d^5y \, \omega(y)$$

is the parity $P: \vec{x} \to -\vec{x}$, $t \to t$, $U \to U^\dagger$, which is the symmetry of QCD.

- We argued that the WZW action is invariant under small deformations of $U(y)$.

However, we can change $U(y)$ discontinuously so that the boundary $U(x)$ is unchanged.

If B_5 is the ball with S_4 as

boundary, take B_5' as the rest of 5d space.

So we could have written

$$S = -n \int_{B_5'} d^5y \, w(y)$$

with a minus sign since the S_4 boundary has opposite orientation.

Both options should give the same path integral for the same goldstone fields.

Of course $n=0$ is a solution.

However, since they appear as e^{iS}, they can differ by 2π, so

$$n \int_{S_5} d^5y \, w(y) = 2\pi \cdot \text{integer}$$

It turns out that the integral over the

five-sphere, with that normalization, gives 2π.

It follows that n must be an integer.

- Gauging the WZW term is nontrivial, since the action is either not explicitly invariant under the global symm, or not explicitly 4-dimensional. See DGH or Witten's paper to see how to add e.m. This leads to $\pi^0 \to \gamma\gamma$ and $\gamma \to \pi\pi\pi$. All are given in terms of the same integer n.

By matching with the anomaly, one gets $n = N_c$, the number of colors of the gauge theory.